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Lie symmetries of nonlinear multidimensional reaction–diffusion systems: II

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Abstract

We present a complete description of the classical (Lie) symmetries of a coupled system of partial differential equations comprising a pair of semilinear reaction–diffusion equations with constant diffusivities and arbitrary nonlinearities in the reaction terms, in any number of spatial dimensions. Part I (Cherniha R M and King J R 2000 *J. Phys. A: Math. Gen.* **33** 267–82, 7839–41) addressed the case of unequal diffusivities; here we complete the analysis by treating the case of equal diffusivities in which the symmetry structure is richer still. Such models arise in the description of numerous physical, chemical and biological systems and we also indicate the possible application in such contexts of some of the specific cases arising from the group classification. Specifically, a variety of Lie's ansätze and exact solutions of the so-called $\lambda - \omega$ reaction–diffusion systems, of a type that arises in mathematical biology, are constructed.

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1. Introduction

This paper, which concludes the analysis commenced in [1], is devoted to consideration of nonlinear reaction–diffusion systems of the form

$$\begin{cases}\lambda_1 U_t = \Delta U + F(U, V)\\ \lambda_2 V_t = \Delta V + G(U, V)\end{cases} \tag{1}$$

in the case $\lambda_1 = \lambda_2 \neq 0$. Without losing generality, we can set $\lambda_1 = 1$, so we consider

$$\begin{cases}U_t = \Delta U + F(U, V)\\ V_t = \Delta V + G(U, V).\end{cases} \tag{2}$$

Here F and G are arbitrary smooth functions, $U = U(t, x)$, $V = V(t, x)$ are unknown functions of $n + 1$ variables t, $x = (x_1, \ldots, x_n)$, Δ is the Laplacian, and the t subscript to the functions U and V denotes differentiation with respect to this variable.

In section 2, the classical Lie scheme is applied to find all possible Lie symmetries which the system (2) can admit. The main results of this section are presented in tables 1–4. In particular, it is established that, among the class (1), the special case (2) is highly non-generic from the algebraic-theoretical point of view. Thus, in contrast to systems of the form (1) with $\lambda_1 \neq \lambda_2$, there are many pairs of nonlinearities (F, G) leading to entirely new types of Lie symmetries in the case $\lambda_1 = \lambda_2$.

In section 3, (1 + 1)-dimensional $\lambda - \omega$ reaction–diffusion systems (see [2], for example, for background) of the form

$$\begin{cases} U_t = \Delta U + \rho^{\alpha_0}(\beta_1 U - \beta_2 V) + \beta_{10} U - \beta_{20} V \\ V_t = \Delta V + \rho^{\alpha_0}(\beta_2 U + \beta_1 V) + \beta_{20} U + \beta_{10} V \end{cases} \tag{3}$$

are considered. Here $\alpha_0, \beta_k, \beta_{k0}, k = 1, 2$, are real parameters and $\rho^2 = U^2 + V^2$. This system with $\alpha_0 = 2$ has been widely studied using qualitative and numerical methods (see, for example, [3], [4] and [2], chapter 12). The results of section 2 establish that this system admits a non-trivial Lie symmetry when $\beta_{10} = 0$. All nonequivalent Lie ansätze are presented, together with formulae for the generation of new solutions from existing ones and examples of exact solutions.

Finally, in section 4 discussion of the results obtained here and in [1] is presented.

2. Lie symmetries of system (2)

Hereinafter the invariance algebra $AE(1.n)$ generated by the operators P_a, J_{ab} and P_t (defined in [1]) is again called *the trivial Lie algebra* of the systems under consideration. Thus, we aim to find all pairs of functions (F, G) that lead to extensions of the trivial Lie algebra of the nonlinear system (2).

It should be stressed that for many reaction–diffusion (RD) systems of the form (2), the relevant Lie algebras can be obtained from those for $\lambda_1 \neq \lambda_2 \neq 0$ (see tables 1, 3–6 in [1]) by formally setting $\lambda_1 = \lambda_2 = 1$. However, we also have the following: (a) cases 3, 4 and 9, table 1, case 15, table 3, case 9, table 5 and case 9, table 6 of [1] have to be considered separately for $\lambda_1 = \lambda_2 = 1$ because the relevant nonlinearities and symmetry operators contain the factors $(\lambda_1 - \lambda_2)^{-1}$ and/or $\lambda_1 - \lambda_2$; (b) there are several systems arising in tables 1–6 of [1] which admit *additional* operators of Lie symmetry when $\lambda_1 = \lambda_2 = 1$ and such systems therefore need to be listed again here.

Now we remind the reader that the most general form of the infinitesimal operator

$$X = \xi^0(t, x, U, V)\partial_t + \xi^a(t, x, U, V)\partial_{x_a} + \eta^U(t, x, U, V)\partial_U + \eta^V(t, x, U, V)\partial_V \tag{4}$$

generating Lie symmetries of the system (2) is given by the following coefficients [1]:

$$\begin{cases} \xi^0 = 2A(t) \\ \xi^a = c_{ab}x_b + \dot{A}(t)x_a + g_a(t) \qquad a, b = 1, \ldots, n \quad a \neq b \\ \eta^U = -\frac{1}{2}\left(\frac{1}{2}|x|^2\ddot{A}(t) + \dot{g}_a(t)x_a\right)U + r^1(t)U + q^1(t)V + P^1(t, x) \\ \eta^V = -\frac{1}{2}\left(\frac{1}{2}|x|^2\ddot{A}(t) + \dot{g}_a(t)x_a\right)V + r^2(t)V + q^2(t)U + P^2(t, x) \end{cases} \tag{5}$$

where $A(t), g_a(t), a = 1, \ldots, n, r^k(t), q^k(t), P^k(t, x), k = 1, 2$ are smooth functions that need to be determined, $c_{ab} + c_{ba} = 0, c_{ab} \in \mathbb{R}$, and the dots over the functions denote differentiation with respect to the variable t. Simultaneously, these coefficients must satisfy the so-called classification equations

$$\begin{cases} \frac{\partial \eta^U}{\partial t} - \Delta\eta^U + F\left(\frac{\partial \eta^U}{\partial U} - \frac{\partial \xi^0}{\partial t}\right) + G\frac{\partial \eta^U}{\partial V} = \eta^U\frac{\partial F}{\partial U} + \eta^V\frac{\partial F}{\partial V} \\ \frac{\partial \eta^V}{\partial t} - \Delta\eta^V + G\left(\frac{\partial \eta^V}{\partial V} - \frac{\partial \xi^0}{\partial t}\right) + F\frac{\partial \eta^V}{\partial U} = \eta^U\frac{\partial G}{\partial U} + \eta^V\frac{\partial G}{\partial V}. \end{cases} \tag{6}$$

To find all pairs of (F, G) for which the system (2) has a non-trivial Lie symmetry, we have to construct all non-equivalent solutions of the classification equations (6). It should be stressed that this is a highly non-trivial problem because (6) are not linear partial differential equations (PDEs) with respect to the U and V in the usual sense, but PDEs with several additional unknown functions $A(t), g_a(t), a = 1 \ldots n, r^k(t), q^k(t), P^k(t, x), k = 1, 2$ that play the role of parameters and have to be determined. Only the cases (ii) $q^1(t) = 0, q^2(t) \neq 0$ and (iii) $q^1(t)q^2(t) \neq 0$ need to be considered because the case $q^1(t) \neq 0, q^2(t) = 0$ is equivalent to the case (ii) and the results for the case (i) $q^1(t) = q^2(t) = 0$ turn out to follow formally from those in [1] in the manner noted above.

Note that we have listed only locally non-equivalent systems with respect to the relevant local substitutions, having the form

$$\begin{cases} U \to c_{11}U + (c_{12} + c_1t)V + c_3\exp(c_{13}t)U + c_5t + c_{10} \\ V \to (c_{21} + c_2t)U + c_{22}V + c_4\exp(c_{23}t)V + c_6t + c_{20} \end{cases} \tag{7}$$

where the coefficients c with subscripts are determined by the form of the system in question (see section 4 for further discussion). One can see that the set of substitutions (7) is much wider than in the case $\lambda_1 \neq \lambda_2 \neq 0$ (see (19) in [1]) because formula (7) contains the terms $(c_{12} + c_1t)V$ and $(c_{21} + c_3t)U$. It should be stressed that the set of substitutions (7) contains the so-called equivalence transformations (in the sense of [5], subsection 2.6) of (2) as a particular case for $c_1 = c_2 = \cdots = c_6$. We construct this set of substitutions instead of equivalence transformations because it gives a much wider range of possibilities for reduction of the number of RD systems (2) with the same Lie symmetry (see examples in section 4).

Now let us state a theorem which gives complete information on the classical symmetries of the system (2) under the additional condition that the system contains at least one operator (4) and (5) with $q^2(t) \neq 0$.

Theorem 1. *All possible maximal algebras of invariance (MAI) of the nonlinear system (2) for any fixed pair of functions (F, G) are presented in tables 1–4. Any other system of the form (2) with non-trivial Lie symmetry can be reduced by a local substitution of the form (7) to one of those given in the tables.*

Sketch of the proof of theorem 1. The proof is similar to that given in [1], so we present only the main steps here.

Taking into account (5), one sees that the most non-trivial symmetry can occur when

$$E \equiv \tfrac{1}{2}|x|^2\ddot{A}(t) + \dot{g}_a(t)x_a \neq 0. \tag{8}$$

Substituting coefficients (5) into (6) and solving the system obtained using the restriction (8), it can be established that the most general forms of such systems coincide with those presented in [1] (see table 1, cases 1 and 6, with $\lambda_1 = \lambda_2 = 1$) and contain arbitrary smooth functions $f(V/U)$ and $g(V/U)$. In other words, we obtain either Galilei-invariant systems or pseudo-Galilean-invariant systems with the same representation. However, finding all systems with possible extensions of these two invariance algebras, we arrive at several new RD systems of the form (2) with Lie symmetries which do not occur in the case $\lambda_1 \neq \lambda_2 \neq 0$. These systems and the corresponding Lie symmetries are listed in table 1.

Table 1. Galilei-invariant and pseudo-Galilean-invariant systems of the form (2).

Sl no	Nonlinearities (F, G)	Restrictions	Basic operators of MAI
1	$\beta_1 V$ $\beta_2 \frac{V^2}{U}$	$\beta_2 \neq \beta_1$	$AE(1.n)$, $Q = U\partial_U + V\partial_V$ $G_a = tP_a - \frac{x_a}{2}Q, a = 1, \ldots, n$ $D_{02} = D_{00} - 2V\partial_V$ $\Pi = tD_{02} - t^2\partial_t - \frac{1}{4}\|x\|^2 Q$ $-t\left(\frac{n}{2} + \frac{\beta_1}{\beta_2-\beta_1}\right)Q + \frac{1}{\beta_1-\beta_2}U\partial_V$
2	$\beta_1 V$ $\beta_1 \frac{V^2}{U}$	$\beta_1 \neq 0$	$AE(1.n)$, Q, $G_a, a = 1, \ldots, n$ D_{02}, $Y^1_{\beta t} = \beta_1 tQ + U\partial_V$
3	$\beta_1 V$ $\beta_1 \frac{V^2}{U} + \beta_{21}U$	$\beta_1\beta_{21} \neq 0$	$AE(1.n)$, Q, $G_a, a = 1, \ldots, n$, $Y^1_{\beta t}$
4	$\beta_1 V$ $\beta_1 \frac{V^2}{U} + \beta_{21}U + \alpha_0 V$	$\beta_1\alpha_0 \neq 0$	Q, $G_a, a = 1, \ldots, n$, $\mathcal{Q}^1_{\alpha_0} = \exp(\alpha_0 t)(\beta_1 Q + \alpha_0 U\partial_V)$
5	$\beta_{10}U + \beta_1 V + \beta_0 U\log U$ $\beta_{10}V + \beta_{21}U + \beta_1\frac{V^2}{U} + \beta_0 V\log U$	$\beta_0\beta_1 \neq 0$	$AE(1.n)$, $\mathcal{Q}_{\beta_0} = \exp(\beta_0 t)Q$ $\mathcal{G}_a = \exp(\beta_0 t)\left(\partial_a - \frac{\beta_0}{2}x_a Q\right)$ $a = 1, \ldots, n$, $Q^1_\gamma at\gamma = -\frac{\beta_1}{\beta_0}$
6	$\beta_{10}U + \beta_1 V + \beta_0 U\log U$ $\beta_{20}V + \beta_{21}U + \beta_1\frac{V^2}{U} + \beta_0 V\log U$	$\beta_0\beta_1 \neq 0$ $\beta_{20} - \beta_{10} = \beta_0$	$AE(1.n)$, $\mathcal{Q}_{\beta_0}$, $\mathcal{G}_a, a = 1, \ldots, n$ $\mathcal{Y}^1_{\beta t} = \exp(\beta_0 t)Y^1_{\beta t}$
7	$\beta_{10}U + \beta_1 V + \beta_0 U\log U$ $\beta_{20}V + \beta_{21}U + \beta_1\frac{V^2}{U} + \beta_0 V\log U$	$\beta_0\beta_1 \neq 0$ $\beta_{20} - \beta_{10} \neq \beta_0$ $\beta_{20} \neq \beta_{10}$	$AE(1.n)$, $\mathcal{Q}_{\beta_0}$, $\mathcal{G}_a, a = 1, \ldots, n$ $\exp[(\beta_{20} - \beta_{10})t]\times$ $[\beta_1 Q + (\beta_{20} - \beta_{10} - \beta_0)U\partial_V]$
8	$\beta_{10}U + \beta_0 U\log U$ $\beta_{10}V + \beta_{21}U + \beta_0 V\log U$	$\beta_0 \neq 0$	$AE(1.n)$, $\mathcal{Q}^1_0$ $\mathcal{Q}_{\beta_0}$, $\mathcal{G}_a, a = 1, \ldots, n$ $R^1_{\beta t}$ with $\beta = \beta_{21}$
9	$\beta_{10}U + \beta_0 U\log U$ $\beta_{20}V + \beta_{21}U + \beta_0 V\log U$	$\beta_0 \neq 0$ $\beta_{10} \neq \beta_{20}$	$AE(1.n)$, $\mathcal{Q}^1_0$ with $\alpha = \beta_{20} - \beta_{10}$ $\mathcal{Q}_{\beta_0}$, $\mathcal{G}_a, a = 1, \ldots, n$ $\left(V + \frac{\beta_{21}}{\beta_{20}-\beta_{10}}U\right)\partial_V$
10	$\beta_1 U\exp\left(-\gamma_0\frac{V}{U}\right)$ $(\beta_2 U + \beta_1 V)\exp\left(-\gamma_0\frac{V}{U}\right)$	$\gamma_0 \neq 0$	$AE(1.n)$, $Q = U\partial_U + V\partial_V$ $G_a = tP_a - \frac{x_a}{2}Q, a = 1, \ldots, n$ $D^1_{\gamma_0} = D_{00} + \frac{2}{\gamma_0}U\partial_V$
11	$\exp(-\gamma_0 W)(\beta_1 U + \beta_2 V)$ $\exp(-\gamma_0 W)(\beta_1 V - \beta_2 U)$	$\gamma_0 \neq 0$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, Q $G_a, a = 1, \ldots, n$ $D^{12}_{\gamma_0} = D_{00} + \frac{2}{\gamma_0}(U\partial_V - V\partial_U)$
12	$\gamma_1 UW + \beta_{10}U + \beta_{20}V$ $\gamma_1 VW + \beta_{10}V - \beta_{20}U$	$\gamma_1 \neq 0$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, Q, $G_a, a = 1, \ldots, n$ $Y^{12} = \gamma_1 tQ + (U\partial_V - V\partial_U)$
13	$(\gamma_1 U + \gamma_2 V)W +$ $\beta_{10}U + \beta_{20}V$ $(\gamma_1 V - \gamma_2 U)W +$ $\beta_{10}V - \beta_{20}U$	$\gamma_2 \neq 0$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, Q, $G_a, a = 1, \ldots, n$ $\mathcal{Q}^{12}_{\gamma_2} = \exp(-\gamma_2 t)[\gamma_1 Q$ $-\gamma_2(U\partial_V - V\partial_U)]$
14	$\beta_0 U\log\rho + \gamma_1 UW +$ $\beta_{10}U + \beta_{20}V$ $\beta_0 V\log\rho + \gamma_1 VW +$ $\beta_{10}V - \beta_{20}U$	$\beta_0 \neq 0$ $\rho^2 = U^2 + V^2$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, $\mathcal{Q}_{\beta_0}$ $\mathcal{G}_a, a = 1, \ldots, n$ $Q^{12}_{\gamma_1\beta_0} = \gamma_1 Q + \beta_0(U\partial_V - V\partial_U)$

Table 1. (Continued.)

Sl no	Nonlinearities (F, G)	Restrictions	Basic operators of MAI
15	$\beta_0 U \log\rho + (\gamma_1 U + \gamma_2 V)W + \beta_{10}U + \beta_{20}V$ $\beta_0 V \log\rho + (\gamma_1 V - \gamma_2 U)W + \beta_{10}V - \beta_{20}U$	$\beta_0 \neq 0$ $\beta_0 \neq -\gamma_2$ $\rho^2 = U^2 + V^2$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, $\mathcal{Q}_{\beta_0}$, $\mathcal{G}_a$, $a = 1, \ldots, n$ $\mathcal{Q}^{12}_{\gamma_2\beta_0} = \exp(-\gamma_2 t)[\gamma_1 Q - (\beta_0 + \gamma_2)(U\partial_V - V\partial_U)]$
16	$\beta_0 U \log\rho + (\gamma_1 U + \gamma_2 V)W + \beta_{10}U + \beta_{20}V$ $\beta_0 V \log\rho + (\gamma_1 V - \gamma_2 U)W + \beta_{10}V - \beta_{20}U$	$\beta_0 \neq 0$ $\beta_0 = -\gamma_2$ $\rho^2 = U^2 + V^2$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, $\mathcal{Q}_{\beta_0}$, $\mathcal{G}_a$, $a = 1, \ldots, n$ $\mathcal{Y}^{12} = \exp(\beta_0 t)[\gamma_1 t Q + (U\partial_V - V\partial_U)]$

Let us clarify how all possible extensions of the Lie symmetry of the Galilei-invariant system

$$\begin{cases} U_t = \Delta U + Uf(\omega) \\ V_t = \Delta V + Vg(\omega) \qquad \omega = V/U \end{cases} \tag{9}$$

were obtained (see nos 1–4 and 10–13 in table 1). Substituting coefficients (5) and the functions $F = Uf, G = Vg$ into (6), we obtain two expressions which form an ODE system for the functions $f(\omega)$ and $g(\omega)$ depending on the variable ω and the functions $A, r^k, q^k, P^k, k = 1, 2$ depending on the variable t (the system is omitted here because of its awkwardness). Since system (9) is invariant under the Galilei algebra, the two expressions obtained are identically satisfied by the coefficients (5) for

$$\begin{aligned} &A = d_0 \qquad g_a = g_{a0}t + d_a \qquad a = 1, 2, \ldots, n \\ &r^k = r_0 \qquad q^k = P^k = 0 \qquad k = 1, 2 \end{aligned} \tag{10}$$

(hereafter $d_0, r_0, d_a, g_{a0}, a = 1, 2, \ldots, n$ are arbitrary parameters corresponding to the operators $P_t, Q, P_a, G_a, a = 1, 2, \ldots, n$, respectively). To find all possible systems with a wider Lie symmetry, we have to obtain the solutions to the two expressions which involve a more general form of functions $A, g_a, a = 1, 2, \ldots, n, r^k, q^k, P^k, k = 1, 2$ than (10). In the general case we can split them into separate parts for the terms in $|x|^2$ and $x_a, a = 1, 2, \ldots, n$ (see η^U, η^V in (5)); we thus find the most general forms of the functions to be

$$\begin{aligned} &A(t) = \tfrac{1}{2}et^2 + A_1 t + d_0 \qquad g_a = g_{a0}t + d_a \qquad a = 1, 2, \ldots, n \\ &P^k(t, x) = p^k(t) \qquad k = 1, 2 \end{aligned} \tag{11}$$

where e and A_1 are arbitrary parameters, $p^1(t)$ and $p^2(t)$ are arbitrary smooth functions. Simultaneously, we obtain the system of equations

$$\begin{cases} \dot r^1 + \dot q^1\omega + \frac{ne}{2} + \frac{1}{U}\dot p^1 + (r^1 - (2et + 2A_1))f + q^1\omega g \\ \quad = \left(r^1 + q^1\omega + \frac{1}{U}p^1\right)(f - \omega f_\omega) + \left(r^2\omega + q^2 + \frac{1}{U}p^2\right) f_\omega \\ \dot r^2 + \frac{\dot q^2}{\omega} + \frac{ne}{2} + \frac{1}{V}\dot p^2 + (r^2 - (2et + 2A_1))g + \frac{q^2}{\omega} f \\ \quad = \left(r^2 + \frac{q^2}{\omega} + \frac{1}{V}p^2\right)(g + \omega g_\omega) - \left(r^1\omega + q^1\omega^2 + \frac{1}{V}p^1\omega^2\right) g_\omega. \end{cases} \tag{12}$$

Since the unknown functions $f(\omega)$ and $g(\omega)$ do not depend separately on U and V, one can again split the system (12) into separate parts, for $\frac{1}{U}$ and $\frac{1}{V}$, and obtain the following systems of ODEs

$$\begin{cases} (p^2 - \omega p^1) f_\omega + p^1 f = \dot p^1 \\ (p^2 - \omega p^1)\omega g_\omega + p^2 g = \dot p^2 \end{cases} \tag{13}$$

$$\begin{cases} (q^2 - q^1\omega^2 + (r^2 - r^1)\omega) f_\omega + (2et + 2A_1 + q^1\omega) f - q^1\omega g = \dot{r}^1 + \dot{q}^1\omega + \frac{ne}{2} \\ (q^2 - q^1\omega^2 + (r^2 - r^1)\omega) g_\omega + \left(2et + 2A_1 + \frac{q^2}{\omega}\right) g - \frac{q^2}{\omega} f = \dot{r}^2 + \frac{\dot{q}^2}{\omega} + \frac{ne}{2} \end{cases} \tag{14}$$

for the functions $f(\omega)$, $g(\omega)$ and $r^k(t)$, $q^k(t)$, $p^k(t)$, $k = 1, 2$.

It can be seen that system (13) is identically satisfied by $p^1 = p^2 = 0$. On the other hand, solving the subsystem in the case $p^1 \neq 0$ or $p^2 \neq 0$, we arrive at the functions

$$f = \alpha_{10} + \alpha_{11}\omega \qquad g = \alpha_{20} + \alpha_{22}\omega^{-1} \tag{15}$$

where $\alpha_{k0}, \alpha_{kk}, k = 1, 2$ are constants. Obviously, substituting (15) into (9) will result in a *linear* RD system. So, we must put $p^1 = p^2 = 0$ and then only system (14) needs to be solved. In a natural way the three distinct cases mentioned above appear: (i) $q^1 = q^2 = 0$; (ii) $q^1 = 0, q^2 \neq 0$; (iii) $q^1 q^2 \neq 0$. In case (i), system (14) is not coupled, therefore its solution is quite similar to that for the general case $\lambda_1 \neq \lambda_2, \lambda_1\lambda_2 \neq 0$ [1]. The relevant computations were done for $\lambda_1 = \lambda_2 = 1$ and the systems and the relevant MAI (see cases 1, 2, 4 and 5 in table 1 of [1]) were found thereby. We have established that the operator Y (see case 4) takes a distinct form, namely $Y = U\partial_U - V\partial_V - 2\beta_0 t(U\partial_U + V\partial_V)$.

Consider now the second case. Since $q^1 = 0$, the first equation of the system (14) can be separately integrated and one finds

$$f(\omega) = \begin{cases} \frac{\dot{r}^1}{q^2}\omega + c_0 & e = A_1 = 0 \\ \beta_1 \exp\left(-2\frac{et+A_1}{q^2}\omega\right) + \frac{\dot{r}^1 + ne/2}{2(et+A_1)} & et + A_1 \neq 0 \end{cases} \tag{16}$$

if $r^1 = r^2$, and

$$f(\omega) = \begin{cases} \frac{\dot{r}^1}{r^2 - r^1}\log\left|\omega + \frac{q^2}{r^2 - r^1}\right| + c_0 & e = A_1 = 0 \\ \beta_1\left(\omega + \frac{q^2}{r^2 - r^1}\right)^{-d} + \frac{\dot{r}^1 + ne/2}{2(et+A_1)} & d = \frac{2(et+A_1)}{r^2 - r^1} \neq 0 \end{cases} \tag{17}$$

if $r^1 \neq r^2$ (hereafter $c_0, \beta_1, 0 \neq d$ are arbitrary constants).

It can be observed that the system (9) with an arbitrary function with a constant term, i.e. $f(\omega) = f_0(\omega) + c_0$, is reduced to the same one with $c_0 = 0$ using the local substitution

$$U \to \exp(c_0 t)U \qquad V \to \exp(c_0 t)V. \tag{18}$$

So, without losing generality, we will assume that c_0 or its analogues vanish in (16) and (17).

Taking into account the substitution noted above, we obtain $f = \beta_1\omega, \beta_1 = \frac{\dot{r}^1}{q^2} \neq 0$ (see (16) for $e = A_1 = 0$). A special case $\beta_1 = 0$ leads to a *linear* RD system so that we do not consider it. Substituting the function f into the second equation of (14), we arrive at the equation

$$g_\omega + \frac{1}{\omega}g = 2\beta_1 + \frac{\ddot{r}^1}{\dot{r}^1}\frac{1}{\omega}. \tag{19}$$

Since the left-hand side of (19) does not depend on the variable t, there are only the following two possibilities

$$r^1 = \begin{cases} r_0 \exp(\alpha_0 t) + r_{00} & \alpha_0 \neq 0 \\ r_0 t + r_{00} & \alpha_0 = 0 \end{cases} \tag{20}$$

for the function $r^1(t)$ in the right-hand side of (19). Having (20), one finds the general solution of (19) in the form

$$g = \beta_1\omega + \beta_{21}\omega^{-1} + \alpha_0 \tag{21}$$

(in formulae (19) and (21) $r_0, r_{00}, \alpha_0, \beta_1, \beta_{21}$ are arbitrary constants).

Substituting the found functions f and g into the RD system (9) and $r^1 = r^2, q^1 = 0$, $q^2 = \frac{\dot{r}^1}{\beta_1}$ into (4) and (5), the systems and the relevant MAI listed in cases 3 and 4, table 1, are obtained.

Considering (16) for $et + A_1 \neq 0$, one obtains

$$f(\omega) = \beta_1 \exp(-\gamma_0\omega) \qquad \frac{et + A_1}{q^2} = \gamma_0 \neq 0 \qquad \dot{r}^1 + ne/2 = 0. \tag{22}$$

Substituting the function f from (22) into the second equation of (14), we arrive at the equation

$$g_\omega + \left(\gamma_0 + \frac{1}{\omega}\right) g = \frac{\beta_1}{\omega} \exp(-\gamma_0\omega) + \frac{e}{(et + A_1)\omega}. \tag{23}$$

The left-hand side of (19) again does not depend on the variable t, therefore the restriction $e = 0$ springs up which leads immediately to $r^1 = r^2 = r_0$, $q^2 = 2A_1/\gamma_0$. We can now easily solve the equation and find the function

$$g(\omega) = \exp(-\gamma_0\omega)\left(\beta_1 + \frac{\beta_2}{\omega}\right) \tag{24}$$

where $r_0, \gamma_0, A_1, \beta_1, \beta_2$ are arbitrary constants. Substituting the found functions f and g into the RD system (9) and $r^k, q^k, k = 1, 2$ into (4) and (5), the system and the relevant MAI listed in case 10, table 1, are obtained.

Similarly, the formula (17) for $e = A_1 = 0$ has been analysed and the functions

$$\begin{aligned} f(\omega) &= \beta_1 \log(\omega + \gamma) \\ g(\omega) &= \left(\beta_2 + \frac{\gamma(\beta_2 - \beta_1)}{\omega}\right) \log(\omega + \gamma) + \frac{\gamma\beta_0}{\omega} + \beta_0 \end{aligned} \tag{25}$$

in which $\gamma \neq 0, \beta_0, \beta_1, \beta_2$ are arbitrary constants, and the relevant operators of MAI were found. However, the RD system (9) with the nonlinearities (25) and the relevant MAI are reduced by the substitution

$$U \to U \qquad V \to V - \gamma U \tag{26}$$

to the systems and MAI listed in cases 4 or 5, table 1 of [1] (at $\lambda_1 = \lambda_2 = 1$) that have been found above in the case (i).

Finally consider (17) at $et + A_1 \neq 0$. It should be stressed that the case $d = -1$ is special because it leads to the linear function

$$f(\omega) = \beta_1\omega + \frac{\beta_1 q^2}{r^2 - r^1} + \frac{\dot{r}^1 + ne/2}{2(et + A_1)}. \tag{27}$$

Again we can suppress the constant term in (27) and, taking into account the expression for $d = -1$ in (17), therefore we find

$$f(\omega) = \beta_1\omega \qquad \beta_1 q^2 = \dot{r}^1 + ne/2 \qquad r^2 - r^1 = -2(et + A_1). \tag{28}$$

Having (28), the second equation of (14) can be reduced to the form

$$g_\omega + \frac{\gamma - \omega}{\omega(\gamma + \omega)} g = \frac{2\beta_1\gamma - 2e(r^2 - r^1)^{-1}}{\gamma + \omega} + \frac{\dot{q}^2(r^2 - r^1)^{-1}}{\omega(\gamma + \omega)} \tag{29}$$

where $\gamma = \gamma(t) = q^2(r^2 - r^1)^{-1}$. This is a linear first-order ODE with respect to the function $g(\omega)$ and its general solution can be easily constructed:

$$g(\omega) = \beta_2\omega + \frac{e + q^2(\beta_2 - \beta_1)}{r^2 - r^1}\left(2 + \frac{\gamma}{\omega}\right) - \frac{\dot{q}^2}{(r^2 - r^1)\omega}. \tag{30}$$

Since the left-hand side of (30) does not depend on the variable t, the expression on the right-hand side must be a function of the variable ω but not of t. It is convenient to consider two cases:

$$e + q^2(\beta_2 - \beta_1) = 0 \tag{31}$$

and

$$e + q^2(\beta_2 - \beta_1) \neq 0. \tag{32}$$

Analysis of the first of them leads to the function $g(\omega) = \beta_2\omega$. Simultaneously, the following functions are obtained:

$$q^2 = \frac{e}{\beta_1 - \beta_2} \qquad r^1 = \left(\frac{\beta_1}{\beta_1 - \beta_2} - \frac{n}{2}\right) et + r_0 \qquad r^2 = r^1 - 2(et + A_1) \tag{33}$$

if $\beta_1 \neq \beta_2$, and

$$q^2 = q_0 \qquad r^1 = \beta_1 q_0 t + r_0 \qquad r^2 = r^1 - 2A_1 \qquad A(t) = A_1 t + d_0 \tag{34}$$

if $\beta_1 = \beta_2$.

Thus, substituting the found functions f and g into the RD system (9) and expressions (33) and (34) into (4) and (5), the systems and the relevant MAI listed in cases 1 and 2, table 1, respectively, are obtained.

Analysis of the second case (see condition (32)) leads to the function

$$g(\omega) = \beta_2\omega + 2\beta_0 + \frac{\beta_0^2}{(\beta_2 - \beta_1)\omega} \tag{35}$$

and the relevant expressions for the functions r^k and q^k, $k = 1, 2$. It turns out that the RD system (9) with the reaction terms (28) and (35) and the relevant MAI are reduced to the same as those for $\beta_0 = 0$, therefore the system and the MAI listed in case 1, table 1, are again obtained. One can easily check this by applying first the substitution (26) for $\gamma = \frac{\beta_0}{\beta_2 - \beta_1}$ and then (18) for $c_0 = \frac{\beta_0\beta_1}{\beta_1 - \beta_2}$. So, condition (32) leads to the RD system and Lie symmetry that are locally equivalent to those found earlier.

To complete the examination of case (ii), formula (17) with $et + A_1 \neq 0$ and $d \neq -1$ should be analysed. After similar computations we obtain

$$f(\omega) = \beta_1(\omega + \gamma)^{-d} \tag{36}$$

$$\gamma = q^2(r^2 - r^1)^{-1} \qquad \dot{r}^1 + ne/2 = 0 \qquad d(r^2 - r^1) = 2(et + A_1) \tag{37}$$

instead of (28). Taking into account formulae (36) and (37), the second equation of (14) can be reduced to the form

$$(\gamma + \omega)g_\omega + \frac{\gamma + d\omega}{\omega} g = \frac{\beta_1\gamma}{\omega(\gamma + \omega)^d} + \frac{\dot{q}^2(r^2 - r^1)^{-1}}{\omega} + \frac{\dot{r}^2 + ne/2}{r^2 - r^1}. \tag{38}$$

In contrast to (29), γ is a constant in (36)–(38) so the condition $e = 0$ and the functions

$$q^2 = \frac{2\gamma A_1}{d} \qquad r^1 = r_0 \qquad r^2 = r^1 + \frac{2A_1}{d} \qquad A(t) = A_1 t + d_0 \tag{39}$$

are obtained. Substituting (39) into (38), we find its general solution:

$$g(\omega) = \frac{\gamma(\beta_2 - \beta_1) + \beta_2\omega}{\omega(\gamma + \omega)^d}. \tag{40}$$

However, the RD system (9) with the nonlinearities (36) and (40) and the relevant MAI are reduced by substitution (26) to the system and MAI listed in case 2, table 1 of [1] (at $\lambda_1 = \lambda_2 = 1$) that have already been found in the case (i).

The investigation of case (ii) is now completed. Note that substitutions (18) and (26) belonging to the set (7) were simultaneously found.

In complete analogy with case (ii), we have analysed the last case, (iii), and found the systems and MAI listed in cases 11–13, table 1. Simultaneously the relevant local substitutions of the form (7) were found.

If the restriction (8) does not apply, i.e. if $E = 0$, then (5) takes the form

$$\begin{cases} \xi^0 = 2A_1 t + d_0 \\ \xi^a = c_{ab}x_b + A_1 x_a + d_a \qquad a, b = 1, \ldots, n \qquad a \neq b \\ \eta^U = r^1(t)U + q^1(t)V + P^1(t, x) \\ \eta^V = r^2(t)V + q^2(t)U + P^2(t, x) \end{cases} \tag{41}$$

where $A_1, d_0, d_1, \ldots, d_n$ are arbitrary parameters. In this case it can be shown that

$$P^k(t, x) = p^k(t) \qquad k = 1, 2 \tag{42}$$

holds for any system of the form (2), except for the system

$$\begin{cases} U_t = \Delta U + f(U) \\ V_t = \Delta V + \beta V + g(U) \end{cases} \tag{43}$$

where f and g are arbitrary functions and $\beta \in \mathbb{R}$. Of course, using the set (7) of local substitutions, this system can be rewritten in other forms but we do not give all the locally equivalent systems only write one of them.

System (43) is invariant with respect to the additional Lie symmetry operator $X^\infty_\beta = P_\beta(t, x)\partial_V$ or $X^\infty_0 = P_0(t, x)\partial_V$ (the functions $P_\beta(t, x)$ and $P_0(t, x)$ are specified in remark 1 below). All possible pairs of functions (f, g) leading to extensions to the Lie symmetry of (43) have been found (see cases 4, 7, 8, 10, 16–18, 20–26 in table 3, and cases 14–24 in table 4).

Consider the general case, i.e. when (2) does not coincide with (43). Substituting (41) and (42) into the classification equations (6) we arrive at

$$\begin{cases} \dot r^1(t)U + \dot q^1(t)V + \dot p^1(t) + F(r^1(t) - 2A_1) + Gq^1(t) \\ \quad = (r^1(t)U + q^1(t)V + p^1(t))\frac{\partial F}{\partial U} + (r^2(t)V + q^2(t)U + p^2(t))\frac{\partial F}{\partial V} \\ \dot r^2(t)V + \dot q^2(t)U + \dot p^2(t) + G(r^2(t) - 2A_1) + Fq^2(t) \\ \quad = (r^1(t)U + q^1(t)V + p^1(t))\frac{\partial G}{\partial U} + (r^2(t)V + q^2(t)U + p^2(t))\frac{\partial G}{\partial V}. \end{cases} \tag{44}$$

To find all pairs of (F, G) for which the system (2) (with structures other than those already found) has a non-trivial Lie symmetry, we have to construct all non-equivalent solutions of the linear system (44), which is much simpler than the system (6) with coefficients (5) because it contains as parameters the functions $r^k(t)$, $q^k(t)$, $p^k(t)$, depending only on one variable.

Consider case (iii), $q^1(t)q^2(t) \neq 0$. It turns out that the system (44) has non-vanishing solutions leading to non-trivial Lie symmetries only in the case $r^1(t) = r^2(t), q^1(t) + q^2(t) = 0$. Taking into account this fact, system (44) has been solved using the 'polar' coordinates $\rho^2 = U^2 + V^2$, $W = \tan^{-1}(V/U)$. All possible pairs of (F, G) and the relevant coefficients $r^k(t), q^k(t), p^k(t), k = 1, 2$ have been found and the results are summarized in table 2.

In case (ii), $q^1(t) = 0, q^2(t) \neq 0$, the first equation of system (44) contains no function G and can be solved independently from the second, so the technique from [1] can be used to find F and the relevant coefficients $q^2(t), r^k(t), p^k(t), k = 1, 2$. Having obtained a list of these functions, the corresponding functions G are found from the second equation. The results are summarized in tables 3 and 4.

The sketch of the proof is now completed.

Table 2. Case (i) $q^1(t)q^2(t) \neq 0$.

Sl no	Nonlinearities (*F, G*)	Restrictions	Basic operators of MAI
1	$\exp(-\gamma_0 W)(Uf(\omega) + Vg(\omega))$ $\exp(-\gamma_0 W)(Vf(\omega) - Ug(\omega))$	$\gamma_0 \neq 0$ $\omega = \rho\exp(-\gamma W)$ $\rho^2 = U^2 + V^2$ $W = \tan^{-1}\left(\frac{V}{U}\right)$	$AE(1.n)$, $D^{12}_{\gamma_0} = D_{00} + \frac{2}{\gamma_0}\times$ $(\gamma U\partial_U + \gamma V\partial_V +$ $U\partial_V - V\partial_U)$
2	$Uf(\omega) + Vg(\omega)$ $Vf(\omega) - Ug(\omega)$	$\omega = \rho\exp(-\gamma W)$ ρ, W—see above	$AE(1.n)$ $Q^{12}_\gamma = \gamma Q + U\partial_V - V\partial_U$
3	$Uf(\omega) + Vg(\omega) +$ $\alpha W(\gamma U - V)$ $Vf(\omega) - Ug(\omega) +$ $\alpha W(\gamma V + U)$	$\alpha \neq 0$ $\omega = \rho\exp(-\gamma W)$ ρ, W—see above	$AE(1.n)$ $\mathcal{Q}^{12}_\gamma = \exp(\alpha t)Q^{12}_\gamma$
4	$\rho^{\alpha_0}\exp(-\gamma\alpha_0 W)(\beta_1 U + \beta_2 V)$ $\rho^{\alpha_0}\exp(-\gamma\alpha_0 W)(\beta_1 V - \beta_2 U)$	$\alpha_0 \neq 0$	$AE(1.n)$, Q^{12}_γ $D_{\alpha_0} = D_{00} - \frac{2}{\alpha_0}(U\partial_U + V\partial_V)$
5	$\rho^{\alpha_0}\exp(-\gamma\alpha_0 W)(\beta_1 U + \beta_2 V) +$ $\beta_{10}U + \beta_{20}V$ $\rho^{\alpha_0}\exp(-\gamma\alpha_0 W)(\beta_1 V - \beta_2 U) +$ $\beta_{10}V - \beta_{20}U$	$\alpha_0\beta_{20} \neq 0$ $\beta_{10} = -\beta_{20}\gamma$	$AE(1.n)$, Q^{12}_γ $D^{12}_\gamma = D_{\alpha_0} - 2\beta_{20}tQ^{12}_\gamma$
6	$(\beta_1 U + \beta_2 V)\log\rho + (\gamma_1 U + \gamma_2 V)W +$ $\beta_{10}U + \beta_{20}V$ $(\beta_1 V - \beta_2 U)\log\rho + (\gamma_1 V - \gamma_2 U)W +$ $\beta_{10}V - \beta_{20}U$	$\beta_2 \neq 0$	$AE(1.n)$ $Y^{12}_k = r_k(t)Q +$ $q_k(t)(U\partial_V - V\partial_U)$

Remark 1. In tables 2–4, $f(\omega)$, $g(\omega)$ are arbitrary smooth functions, $D_{00} \equiv 2t\partial_t + x_a\partial_a$, and $P_0(t,x)$ and $P_\beta(t,x)$ are arbitrary solutions of the linear heat equations

$$P_t = \Delta P \tag{45}$$

and

$$P_t = \Delta P + \beta P \tag{46}$$

the functions $r_k(t)$, $q_k(t)$, $k = 1, 2$ form a fundamental system of solutions of the linear ODE systems (table 2, case 6)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \beta_1 r + \gamma_1 q \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = -\beta_2 r - \gamma_2 q \tag{47}$$

and (table 3, case 11)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \beta_0 r + \beta_1 q \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = -\gamma r + (\beta_{20} - \beta_{10})q. \tag{48}$$

Remark 2. Taking into account the nonlinearities listed above in tables 1–4, we note that all nonlinear systems of the form (2) admitting an MAI generated by the infinitesimal operator (4), (5) with $q^k(t) = 0$, $k = 1, 2$ coincide with the following systems previously found in [1]: 1, 2, 4 (with $Y = U\partial_U - V\partial_V - 2\beta_0 t(U\partial_U + V\partial_V)$, 5, 6, 7 (with $\beta_{10}\beta_{20} \neq 0$), 8, 9 (with $\beta_0 = 0$ and the operator $\exp(\beta t)U\partial_U$ instead of the $\mathcal{Y}$) in table 1; 1, 2 (with $\gamma \neq 1$), 3 (with $\alpha_1 \neq 0$ or $\beta_1 \neq \beta_2$), 5, 7, 8 (with $\gamma \neq 0$), 9, 11, 13, 16 in table 3; 1, 2, 7, 8 (taking into account remark 3), 9, 11 in table 4; 1, 3, 6–8 in table 5; 10 (taking into account remark 4 for $\alpha \neq 0$; 1) and 19 in table 6. In those cases one need only set $\lambda_1 = \lambda_2 = 1$.

Table 3. Case (ii) $q^1(t) = P^1(t,x) = 0, q^2(t) \neq 0$.

Sl no	Nonlinearities (F, G)	Restrictions	Basic operators of MAI
1	$\exp\left(-\gamma_0 \frac{V}{U}\right) Uf(\omega)$ $\exp\left(-\gamma_0 \frac{V}{U}\right)(Vf(\omega) + Ug(\omega))$	$\omega = U\exp\left(-\gamma\frac{V}{U}\right)$ $\gamma_0 \neq 0$	$AE(1.n)$, $D^1_{\gamma\gamma_0} = D_{00} +$ $\gamma U\partial_U + \gamma V\partial_V + \frac{2}{\gamma_0}U\partial_V$
2	$Uf(\omega)$ $Vf(\omega) + Ug(\omega)$	$\omega = U\exp\left(-\gamma\frac{V}{U}\right)$	$AE(1.n)$, $Q^1_\gamma = \gamma U\partial_U + \gamma V\partial_V + U\partial_V$
3	$Uf(\omega) + \alpha\gamma V$ $Vf(\omega) + Ug(\omega) +$ $\alpha V\left(1 + \gamma\frac{V}{U}\right)$	$\omega = U\exp\left(-\gamma\frac{V}{U}\right)$ $\alpha \neq 0$	$AE(1.n)$, $\mathcal{Q}^1_{\gamma\alpha} = \exp(\alpha t)Q^1_\gamma$
4	0 $Ug(U)$		$AE(1.n)$, Q^1_0 $X^\infty_0 = P_0(t,x)\partial_V$ $D_{01} = D_{00} + 2V\partial_V$
5	$Uf(U)$ $Vf(U)$		$AE(1.n)$, $Q^1_0 = U\partial_V$ $I = V\partial_V$
6	$Uf(U)$ $\beta U + Vf(U)$	 $\beta \neq 0$	$AE(1.n)$, $Q^1_0 = U\partial_V$ $R^1_{\beta t} = \beta tU\partial_V - V\partial_V$
7	βU $\beta V + Ug(U)$	$\beta \neq 0$	$AE(1.n)$, Q^1_0 $X^\infty_\beta = P_\beta(t,x)\partial_V$
8	$\beta_1 U$ $(\beta_1 + \alpha)V + Ug(U)$	 $\alpha \neq 0$	$AE(1.n)$, $\mathcal{Q}^1_0 = \exp(\alpha t)U\partial_V$ X^∞_β with $\beta = \beta_1 + \alpha$
9	$Uf(U)$ $\beta U + \alpha V + Vf(U)$	 $\alpha\beta \neq 0$	$AE(1.n)$, $\mathcal{Q}^1_0$ $\mathcal{R}^1_{\beta t} = \exp(\alpha t)(\beta tU\partial_V - V\partial_V)$
10	β_1 $\beta_2 V + Ug(U)$	 $\beta_1 \neq 0$	$AE(1.n)$, X^∞_β, $\beta = \beta_2$ $\mathcal{Q}^1_{\beta t} = \exp(\beta_2 t)(U - \beta_1 t)\partial_V$
11	$\beta_{10}U + \beta_1 V + \beta_0 U\log U$ $\beta_{20}V + \beta_{21}U + \beta_1\frac{V^2}{U} +$ $(\beta_0 V + \gamma U)\log U$	$\gamma \neq 0$ $\beta_0 \neq 0$ or $\beta_1 \neq 0$	$AE(1.n)$, $Y^1_k = r_k(t)Q + q_k(t)U\partial_V$
12	$\beta_1 U^{\alpha+1}\exp\left(-\alpha\gamma\frac{V}{U}\right) + \beta_{10}U$ $(\beta_2 U + \beta_1 V)U^\alpha\exp\left(-\alpha\gamma\frac{V}{U}\right) +$ $\beta_{10}V + \frac{\beta_{10}}{\gamma}U$	$\alpha\gamma \neq 0$	$AE(1.n)$, Q^1_γ $D^1_1 = D_{00} + \frac{2}{\alpha\gamma}U\partial_V -$ $2\beta_{10}\alpha tQ^1_\gamma$
13	$\beta_1 U^{\alpha+1}$ $\beta_1 VU^\alpha + \beta_2 U^{\alpha_0} + \beta_{20}U$	$\beta_1\beta_2\alpha \neq 0$ $\alpha_0 - \alpha \neq 0$ $\alpha_0 - 1 \neq 0$	$AE(1.n)$, Q^1_0 $D^1_2 = D_{00} - \frac{2}{\alpha}(U\partial_U +$ $(\alpha_0 - \alpha)V\partial_V) + 2\beta_{20}\frac{\alpha_0 - 1}{\alpha}tU\partial_V$
14	$\beta_1 U^{\alpha+1}$ $\beta_1 VU^\alpha + \beta_{20}U$	$\beta_1 \neq 0$ $\alpha(\alpha - 1) \neq 0$	$AE(1.n)$, Q^1_0 D^1_2 with $\alpha_0 = \alpha$, $R^1_{\beta t}$ with $\beta = \beta_{20}$
15	$\beta_1 U^{\alpha+1}$ $\beta_1 VU^\alpha + \beta_2 U\log U + \beta_{20}U$	$\beta_1\beta_2 \neq 0$ $\alpha \neq 0$	$AE(1.n)$, Q^1_0 $D^1_3 = D_{00} - \frac{2}{\alpha}(U\partial_U +$ $(1 - \alpha)V\partial_V) + 2\beta_2 tU\partial_V$
16	$\beta_1 U^{\alpha+1}$ $U^{\alpha+1}(\beta_2 + \beta_{20}\log U)$	$\beta_1\beta_{20} \neq 0$ $\alpha \neq 0, -1$	$AE(1.n)$, X^∞_0 $D^1_{\gamma\gamma_0}$ at $\gamma = -\frac{2}{\alpha}$, $\gamma_0 = -\frac{\alpha\beta_1}{\beta_{20}}$
17	$\beta_1 U$ $\beta_1 V + \beta_{20}U^\alpha + \beta_{21}U$	$\beta_1\beta_{20} \neq 0$ $\alpha \neq 0, 1, 2$	$AE(1.n)$, Q^1_0 X^∞_β with $\beta = \beta_1$ $Q^1_{\alpha t} = U\partial_U + \alpha V\partial_V + \beta_{21}(1 - \alpha)tU\partial_V$

Table 3. (Continued.)

Sl no	Nonlinearities (F, G)	Restrictions	Basic operators of MAI
18	$\beta_1 U$ $\beta_2 V + \beta_{20} U^\alpha$	$\beta_{20}(\beta_2 - \beta_1) \neq 0$ $\alpha \neq 0, 1, 2$ $\beta_2 \neq \alpha\beta_1$	$AE(1.n)$, Q_0^1 with $\alpha = \beta_2 - \beta_1$ X_β^∞ with $\beta = \beta_2$ $Q_\alpha = U\partial_U + \alpha V\partial_V$
19	$\beta_1 U^2$ $\beta_1 VU$	$\beta_1 \neq 0$	$AE(1.n)$, Q_0^1, $V\partial_V$ $R_{\beta t}^1 + \partial_V$ with $\beta = \beta_1$ $D_{03} = D_{00} - 2U\partial_U$
20	0 $\beta_{20} U^\alpha$	$\beta_{20} \neq 0$ $\alpha \neq 0, 1, 2$	$AE(1.n)$, Q_0^1 X_0^∞, D_{01}, Q_α
21	β_1 $\beta_{20} U^\alpha + \beta_{21} U$	$\beta_1\beta_{20} \neq 0$ $\alpha \neq 0, 1, 2$	$AE(1.n)$, X_0^∞ $Q_{\beta t}^1 = (U - \beta_1 t)\partial_V$ $D_5^1 = D_{01} + 2Q_{\alpha t}^1 - \beta_1\beta_{21}(1-\alpha)t^2\partial_V$
22	$\beta_{10} U$ $\beta_{10} V + \beta_{21} U + \gamma U \log U$	$\gamma \neq 0$	$AE(1.n)$, X_β^∞ with $\beta = \beta_{10}$ Q_0^1, $Q + \gamma t U\partial_V$ $D_6^1 = D_{01} + 2\beta_{10} t Q + \gamma\beta_{10} t^2 U\partial_V$
23	$\beta_{10} U$ $\beta_{20} V + \beta_{21} U + \gamma U \log U$	$\gamma \neq 0$ $\beta_{10} \neq \beta_{20}$	$AE(1.n)$, X_β^∞ with $\beta = \beta_{20}$ Q_0^1 with $\alpha = \beta_{20} - \beta_{10}$ Q_γ^1 with $\gamma \to \frac{\beta_{10}-\beta_{20}}{\gamma}$
24	0 $\beta_{20} V + \beta_{21} U + \gamma \log U$	$\beta_{20}\gamma \neq 0$	$AE(1.n)$, X_β^∞ with $\beta = \beta_{20}$ Q_0^1 with $\alpha = \beta_{20}$ $\beta_{20} U\partial_U - (\beta_{21} U + \gamma)\partial_V$
25	0 $\gamma \log U$	$\gamma \neq 0$	$AE(1.n)$, X_0^∞, Q_0^1 D_{01}, $U\partial_U + \gamma t\partial_V$
26	β_1 $\beta_{21} U + \gamma \log U$	$\gamma\beta_1 \neq 0$	$AE(1.n)$, X_0^∞, $Q_{\beta t}^1$ $D_7^1 = D_{00} + 2Q +$ $2\beta_{21} t U\partial_V + (2\gamma t - \beta_{21}\beta_1 t^2)\partial_V$

Remark 3. We have found that case 9 in table 1 of [1] is also valid for $\beta_0 = 0$ while case 8 in table 4 of [4] admits the following generalization (see also [7]): the system

$$\begin{cases} \lambda_1 U_t = \Delta U + U(\beta_1 + \beta_{10}\log U + \gamma_1 \log V) \\ \lambda_2 V_t = \Delta V + V(\beta_2 + \beta_{20}\log V + \gamma_2 \log U) \end{cases} \tag{49}$$

where the arbitrary real parameters $\beta_k, \beta_{k0}, \gamma_k, k = 1, 2$ satisfy $\lambda_2(\beta_{10} + \gamma_1\lambda_2/\lambda_1) \neq \lambda_1(\beta_{20} + \gamma_2\lambda_1/\lambda_2)$ and $\lambda_1\lambda_2 \neq 0$, is invariant with respect to the MAI $\{AE(1.n), X_1, X_2\}$. Here the operators X_1, X_2 are given by $X_k = r_k(t)U\partial_U + q_k(t)V\partial_V, k = 1, 2$, where the functions $r_k(t), q_k(t), k = 1, 2$ form a fundamental system of solutions of the linear ODE system

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \beta_{10} r + \gamma_1 q \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = \beta_{20} q + \gamma_2 r. \tag{50}$$

There are six different forms of solutions of (50), depending on the coefficients γ_k and β_{k0} $(k = 1, 2)$ in the system (49).

Table 4. Case (ii) $q^1(t) = 0$, $P^1(t, x) \neq 0, q^2(t) \neq 0$.

Sl no	Nonlinearities (F, G)	Restrictions	Basic operators of MAI
1	$\exp(-\gamma_0 U) f(\omega)$ $\exp(-\gamma_0 U)(g(\omega) - 2\alpha U f(\omega))$	$\omega = \alpha U^2 + V$ $\alpha\gamma_0 \neq 0$	$AE(1.n)$, $D^1_{\alpha\gamma_0} = D_{00} +$ $\frac{2}{\gamma_0}(\partial_U - 2\alpha U \partial_V)$
2	$f(\omega)$ $g(\omega) - 2\alpha U f(\omega)$	$\omega = \alpha U^2 + V$ $\alpha \neq 0$	$AE(1.n)$, $R^1_\alpha = \partial_U - 2\alpha U \partial_V$
3	$f(\omega) + \gamma U$ $g(\omega) - 2\alpha U f(\omega) + 2\gamma V$	$\omega = \alpha U^2 + V$ $\alpha\gamma \neq 0$	$AE(1.n)$, $\mathcal{R}^1_{\gamma\alpha} = \exp(\gamma t) R^1_\alpha$
4	$\beta_1(\alpha U^2 + V)^{\alpha_0+0.5} - \frac{\beta_{10}}{\alpha_0+0.5}$ $\beta_2(\alpha U^2 + V)^{\alpha_0+1} - 2\alpha U \times$ $\left(\beta_1(\alpha U^2 + V)^{\alpha_0+0.5} - \frac{\beta_{10}}{\alpha_0+0.5}\right)$	$\beta_1 \neq 0$ or $\beta_2 \neq 0$ $\alpha_0 \neq 0, -0.5$ $\alpha \neq 0$	$AE(1.n)$, R^1_α, $D^1_8 = D_{00} -$ $\frac{1}{\alpha_0}\left(U\partial_U + 2V\partial_V + 2\beta_{10} t R^1_\alpha\right)$
5	$\beta_1 + \beta_{10}\log(\alpha U^2 + V)$ $\beta_2\sqrt{\alpha U^2 + V} - 2\alpha U \times$ $(\beta_1 + \beta_{10}\log(\alpha U^2 + V))$	$\beta_{10} \neq 0$ or $\beta_2 \neq 0$ $\alpha \neq 0$	$AE(1.n)$, R^1_α, D^1_8
6	$\beta_1\sqrt{\alpha U^2 + V} - \beta_{10}$ $\beta_2(\alpha U^2 + V) - 2\alpha U \times$ $\times(\beta_1\sqrt{\alpha U^2 + V} - \beta_{10})$	$\alpha\beta_1 \neq 0$	$AE(1.n)$, R^1_α $U\partial_U + 2V\partial_V + \beta_{10} t R^1_\alpha$
7	$\beta_1(\alpha U^2 + V) - \beta_{10}$ $\beta_2(\alpha U^2 + V) - 2\alpha U \times$ $(\beta_1(\alpha U^2 + V) - \beta_{10})$	$\alpha\beta_1\beta_2 \neq 0$	$AE(1.n)$, R^1_α $\exp(\beta_2 t)\left(\beta_1 R^1_\alpha + \beta_2\partial_V\right)$
8	$\beta_1(\alpha U^2 + V)$ $-2\alpha\beta_1 U(\alpha U^2 + V)$	$\alpha\beta_1 \neq 0$	$AE(1.n)$, R^1_α $\beta_1 t R^1_\alpha + \partial_V$
9	$\beta_1\exp(\alpha_0(\alpha U^2 + V)) + \beta_{10}$ $\beta_2\exp(\alpha_0(\alpha U^2 + V)) - 2\alpha U \times$ $(\beta_1\exp(\alpha_0(\alpha U^2 + V)) + \beta_{10})$	$\beta_1 \neq 0$ or $\beta_2 \neq 0$ $\alpha\alpha_0 \neq 0$	$AE(1.n)$, R^1_α $D^1_9 = D_{00} - \frac{2}{\alpha_0}\partial_V + 2\beta_{10} t R^1_\alpha$
10	$\beta_1\sqrt{\alpha U^2 + V} + \gamma U + \beta_{10}$ $\beta_2(\alpha U^2 + V) - 2\alpha U \times$ $(\beta_1\sqrt{\alpha U^2 + V} + \beta_{10}) + 2\gamma V$	$\alpha\beta_1\gamma \neq 0$	$AE(1.n)$, $\mathcal{R}^1_{\gamma\alpha}$ $\gamma(U\partial_U + 2V\partial_V) + \beta_{10} R^1_\alpha$
11	$\beta_1(\alpha U^2 + V) + \gamma U + \beta_{10}$ $\beta_2\alpha U^2 + (\beta_2 + 2\gamma)V - 2\alpha U \times$ $(\beta_1(\alpha U^2 + V) + \beta_{10})$	$\alpha\beta_1\gamma \neq 0$ $\beta_2 \neq -\gamma$ $\beta_2 \neq -2\gamma$	$AE(1.n)$, $\mathcal{R}^1_{\gamma\alpha}$ $\exp((\beta_2 + 2\gamma)t)\left(\frac{\beta_1}{\beta_2+\gamma} R^1_\alpha + \partial_V\right)$
12	$\beta_1(\alpha U^2 + V) + \gamma U + \beta_{10}$ $\gamma(V - \alpha U^2) - 2\alpha U \times$ $(\beta_1(\alpha U^2 + V) + \beta_{10})$	$\alpha\beta_1\gamma \neq 0$	$AE(1.n)$, $\mathcal{R}^1_{\gamma\alpha}$ $\beta_1 t\mathcal{R}^1_{\gamma\alpha} + \exp(\gamma t)\partial_V$
13	$\beta_1(\alpha U^2 + V) + \gamma U + \beta_{10}$ $-2\alpha U(\beta_1(\alpha U^2 + V) + \gamma U + \beta_{10})$	$\alpha\beta_1\gamma \neq 0$	$AE(1.n)$, $\mathcal{R}^1_{\gamma\alpha}$ $\beta_1 R^1_\alpha - \gamma\partial_V$
14	$\beta_1 U$ $\beta_1 V + \beta_{20} U^2 + \beta_{21} U$	$\beta_1\beta_{20} \neq 0$	$AE(1.n)$, Q^1_0, X^∞_β with $\beta = \beta_1$ $Q^1_{\alpha t}$ with $\alpha = 2$, $\exp(\beta_1 t) \times$ $(\beta_1\partial_U + 2\beta_{20} U\partial_V + \beta_1\beta_{21} t\partial_V)$
15	$\beta_1 U$ $\beta_2 V + \beta_{20} U^2$	$\beta_{20} \neq 0$ $\beta_2 \neq \beta_1$ $\beta_2 \neq 2\beta_1$	$AE(1.n)$, Q^1_0 with $\alpha = \beta_2 - \beta_1$ X^∞_β with $\beta = \beta_2$ Q_α with $\alpha = 2$ $\exp(\beta_1 t)\left(\partial_U + \frac{2\beta_{20}}{2\beta_1 - \beta_2} U\partial_V\right)$

Table 4. (Continued.)

Sl no	Nonlinearities (F, G)	Restrictions	Basic operators of MAI
16	$\beta_1 U$ $2\beta_1 V + \beta_{20} U^2$	$\beta_{20}\beta_1 \neq 0$	$AE(1.n)$, Q_0^1 with $\alpha = \beta_1$ Q_α with $\alpha = 2$ $\exp(\beta_1 t)(\beta_1 \partial_U + 2\beta_1\beta_{20} t U \partial_V)$ D_4^1 with $\alpha = 2$ X_β^∞ with $\beta = 2\beta_1$
17	0 $\beta_{20} U^2$	$\beta_{20} \neq 0$	$AE(1.n)$, Q_0^1, X_0^∞ Q_α with $\alpha = 2$, D_{01} $\partial_U + 2\beta_{20} t U \partial_V$
18	0 $\beta_2 V + \beta_{20} U^2$	$\beta_2\beta_{20} \neq 0$	$AE(1.n)$, Q_0^1 with $\alpha = \beta_2$ X_β^∞ with $\beta = \beta_2$, $U\partial_U + 2V\partial_V$ R_α^1 with $\alpha = \frac{\beta_{20}}{\beta_2}$
19	β_1 $\beta_{20} U^2$	$\beta_1\beta_{20} \neq 0$	$AE(1.n)$, $Q_{\beta t}^1$, X_0^∞ $D_{04} = D_{00} + 2U\partial_U + 6V\partial_V$ $\partial_U + 2\beta_{20} t \left(U - \frac{\beta_1}{2} t\right) \partial_V$ $U\partial_U + 2V\partial_V - \beta_1\beta_{20} t^2 \left(U - \frac{\beta_1}{3} t\right) \partial_V - \beta_1 t \partial_U$
20	β_1 $\beta_2 V + \beta_{20} U^2$	$\beta_1\beta_2\beta_{20} \neq 0$	$AE(1.n)$, $\mathcal{Q}_{\beta t}^1 = \exp(\beta_2 t) Q_{\beta t}^1$ X_β^∞ with $\beta = \beta_2$ $\beta_2 \partial_U - 2\beta_{20} \left(U + \frac{\beta_1}{\beta_2}\right) \partial_V$
21	β_1 $\beta_2 V + \beta_{20} \exp U$	$\beta_1 - \beta_2 \neq 0$ $\beta_2 \neq 0$	$AE(1.n)$, $\mathcal{Q}_{\beta t}^1$ X_β^∞ with $\beta = \beta_2$ $\partial_U + V\partial_V$
22	0 $\beta_{20} \exp U$	$\beta_{20} \neq 0$	$AE(1.n)$, Q_0^1, X_0^∞ D_{01}, $\partial_U + V\partial_V$
23	β_1 $\beta_{21} U + \beta_{20} \exp U$	$\beta_1\beta_{20} \neq 0$	$AE(1.n)$, $Q_{\beta t}^1$, X_0^∞ $\partial_U + V\partial_V + \beta_{21} t \left(1 + \frac{\beta_1}{2} t - U\right) \partial_V$
24	$\beta_1 \exp U$ $(\beta_2 - 2\alpha\beta_1 U) \exp U$	$\beta_1\alpha \neq 0$	$AE(1.n)$, X_0^∞ $D_{\alpha\gamma_0}^1$ with $\gamma_0 = -1$

Remark 4. It was found that case 10 in table 6 of [1] admits a generalization by setting $F = \beta_1 \log V$, $G = \beta_2 V^\alpha$ [8], where the constants $\beta_1\beta_2 \neq 0$ and $\alpha \neq 1$. Simultaneously the operator of scale transformations takes the form

$$D_\beta = 2tP_t + x_a P_a + 2\left(U\partial_U + \frac{1}{1-\alpha} V\partial_V\right) + \frac{2\beta_1 t}{\lambda_1(1-\alpha)}\partial_U.$$

Moreover, we have found that case 13 in table 6 of [1] admits a generalization by setting $F = 0$, $G = g(U)$, where $g(U)$ is an arbitrary function.

It is worth commenting on the systems and Lie algebras listed in table 1. One can note that when $\lambda_1 = \lambda_2$ no new representations of the Galilei algebra $AG(1.n)$ and the pseudo-Galilean algebra $A\mathcal{G}(1.n)$ are admitted by system (1). However, there are *new extensions of these algebras* that system (1) with $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$ does not admit. Case 1 (see table 1) represents an RD system that is invariant under the generalized Galilei algebra $AG_2(1.n)$ with a new representation, in that it contains the projective operator Π which cannot be reduced to standard form (cf case 3, table 1 of [1]). Note that this RD system and its MAI were found in [6] for the first time.

Case 2 is a system admitting an absolutely new algebra $AG_2^0(1.n) = \{AG_1(1.n), Y_{\beta t}^1\}$. This algebra and $AG_2(1.n)$ have the same dimensionality but are to be regarded as distinct algebras because the $AG_2^0(1.n)$ algebra contains no projective operator Π.

The systems listed in cases 3, 4, 12 and 13 are invariant under the algebras $\{AG(1.n), Y_{\beta t}^1\}$, $\{AG(1.n), \mathcal{Q}_{\alpha_0}^1\}$, $\{AG(1.n), Y^{12}\}$ and $\{AG(1.n), \mathcal{Q}_{\gamma_2}^{12}\}$, respectively. These algebras are new representations of those listed in cases 4 and 5, table 1 of [1]. Similarly, cases 5–7 represent systems that are invariant under Lie algebras which are new representations of the $A\mathcal{G}(1.n)$ algebra extensions listed in cases 7–9, respectively, of table 1 of [1].

Cases 8 and 9 are systems with absolutely new algebras of Lie symmetries, in that they are extensions of $A\mathcal{G}(1.n)$ by *two* operators. In other words, they represent pseudo-Galilean analogues of $AG_2(1.n)$ and $AG_2^0(1.n)$.

Cases 10 and 11 are systems admitting the extended Galilei algebra $AG_1(1.n)$ with two new representations of the operator of scale transformations.

Finally, cases 14–16 are systems that are invariant under Lie algebras which are new representations of those listed in cases 7–9, respectively, of table 1 of [1].

The nonlinearity V^2/U occurs in a number of the above cases, so it worth noting an (albeit rather contrived) application of such an expression. We focus for definiteness on case 1 of table 1, though others can be derived in a similar framework. Consider a surface on which the proportion of unbound reactions sites, to which a diffusible chemical u can bind reversibly, is θ. A second diffusible chemical v acts as a catalyst in the reaction

$$a + v \to u + v$$

where the concentration A of a further chemical a (with similar notation for other species) is held fixed, while v can undergo a reaction of the form

$$2v \to b$$

at the above-mentioned reactions sites. A simple mathematical model for such a process reads

$$\begin{cases} \theta_t = k_1(1-\theta) - k_2\theta U \\ U_t = \Delta U + k_1(1-\theta) - k_2\theta U + k_3 AV \\ V_t = \Delta V - k_4\theta V^2 \end{cases} \tag{51}$$

where the coefficients $k_1, \ldots, k_4$ are positive constants.

If k_1 and k_2 are sufficiently large then the quasi-steady approximation $\theta = k_1/(k_1 + k_2 U)$ holds and if in addition $U \gg \theta$, the system (51) reduces to

$$\begin{cases} U_t = \Delta U + k_3 AV \\ V_t = \Delta V - \frac{k_1 k_4}{k_1 + k_2 U} V^2. \end{cases} \tag{52}$$

Evidently system (52) is reduced to case 1 in table 1 by the local substitution $k_1 + k_2 U \to U$.

Not surprisingly, a number of the systems are more conveniently expressed in terms of the complex quantity $\Phi = U + iV$ (this is a special feature of dealing with a pair of equations, (2)), so that, for example, table 2, case 5 takes the form of the complex equation

$$\Phi_t = \Delta\Phi + (\beta_1 - \mathrm{i}\beta_2)|\Phi|^{2\alpha_0} \exp(-\gamma\alpha_0 \arg\Phi)\Phi - \beta_{20}(\gamma - \mathrm{i})\Phi. \tag{53}$$

Such equations with $\gamma = 0$ are very familiar in the literature (see the next section for details) and are of interest for their blow-up properties, for example. Other nonlinearities arising in the tables as special from the Lie symmetry point of view are also of interest for other reasons, but we shall not labour the point here.

3. Lie ansätze and solutions of the $\lambda - \omega$ RD system (3)

Consider the $\lambda - \omega$ RD system (3) in (1+1) dimensions, i.e.

$$\begin{cases} U_t = U_{xx} + \rho^{\alpha_0}(\beta_1 U - \beta_2 V) + \beta_{10}U - \beta_{20}V \\ V_t = V_{xx} + \rho^{\alpha_0}(\beta_2 U + \beta_1 V) + \beta_{20}U + \beta_{10}V \end{cases} \tag{54}$$

where $\alpha_0\beta_1 \neq 0$ or $\alpha_0\beta_2 \neq 0$. In a widely used notation (see, e.g. [2], chapter 12) this corresponds to $\lambda(\rho) = \beta_1\rho^{\alpha_0} + \beta_{10}$, $\omega(\rho) = \beta_2\rho^{\alpha_0} + \beta_{20}$, $\rho^2 = U^2 + V^2$. The system (54) with $\alpha_0 = 2$ has been extensively studied (see [2] and papers cited therein) since it is a plausible model for certain biochemical reactions. Here we apply the algebraic-theoretical approach to the investigation of (54).

It follows from theorem 1 and remark 2 that there are three types of Lie algebra that can be admitted by this RD system. The relevant basic operators are (i) $P_t = \partial_t$, $P_x = \partial_x$, $Q_0^{12} = U\partial_V - V\partial_U$ and $D_{\alpha_0} = 2t\partial_t + x\partial_x - \frac{2}{\alpha_0}(U\partial_U + V\partial_V)$, if $\beta_{10} = \beta_{20} = 0$ (see table 2, 4 with $\gamma = 0$); (ii) P_t, P_x, Q_0^{12}, $D_{\alpha_0}^{12} = D_{\alpha_0} - 2\beta_{20}tQ_0^{12}$, if $\beta_{10} = 0$, $\beta_{20} \neq 0$ (see table 2, 5 with $\gamma = 0$); (iii) P_t, P_x, Q_0^{12}, if $\beta_{10} \neq 0$ (see table 2, 2).

We aim to construct the Lie ansätze and seek exact solutions in each of these cases. For such a purpose it is convenient to change the variables (U, V) to the polar variables (ρ, W) defined by

$$U = \rho\cos W \qquad V = \rho\sin W. \tag{55}$$

Substituting (55) into (54) leads to the system

$$\begin{cases} \rho_t = \rho_{xx} - \rho W_x^2 + \beta_{10}\rho + \beta_1\rho^{\alpha_0+1} \\ W_t = W_{xx} + 2\rho^{-1}\rho_x W_x + \beta_{20} + \beta_2\rho^{\alpha_0}. \end{cases} \tag{56}$$

Remark 5. Using the substitution $W \to W + \beta_{20}t$, one can eliminate β_{20} from (56). However, we prefer to keep $\beta_{20} \neq 0$ so the results can be applied more directly (the nonlinearity $\omega(\rho) = \beta_2\rho^{\alpha_0} + \beta_{20}$ is more widely applicable than that with $\beta_{20} = 0$).

It is easily shown that the operators listed above have the form

$$P_t, P_x \qquad Q_0^{12} = \partial_W \qquad D_{\alpha_0}^{12} = D_{\alpha_0} - 2\beta_{20}t\partial_W \tag{57}$$

in polar variables, where $D_{\alpha_0} = 2t\partial_t + x\partial_x - \frac{2}{\alpha_0}\rho\partial_\rho$. According to the usual procedure, to find the similarity reductions it is necessary to solve the Lagrange system

$$\frac{\mathrm{d}t}{\xi^0(t,x)} = \frac{\mathrm{d}x}{\xi^1(t,x)} = \frac{\mathrm{d}\rho}{\eta^1(t,x)\rho} = \frac{\mathrm{d}W}{\eta^2(t,x)} \tag{58}$$

where $\xi^0, \xi^1, \eta^1, \eta^2$ are known coefficients of the infinitesimal operator X, which is given by a linear combination of the relevant operators listed in (57).

In the general case (iii), a full set of non-equivalent Lie ansätze can be constucted by solving (58) for the operators

$$X_1 = P_x + \gamma\partial_W \qquad X_2 = P_t + vP_x + \gamma\partial_W \qquad X_3 = \partial_W \qquad \gamma, v \in R. \tag{59}$$

In the first two cases there is the additional Lie ansatz generated by operators

$$X_4^0 = D_{\alpha_0} + \gamma\partial_W \tag{60}$$

and

$$X_4 = D_{\alpha_0}^{12} + \gamma\partial_W \tag{61}$$

respectively.

Consider the algebra X_1 which generates the ansatz

$$\rho = \varphi(t) \qquad W = \psi(t) + \gamma x. \tag{62}$$

Using (62), we can reduce system (54) to a system of ODEs

$$\begin{cases} \varphi_t = (\beta_{10} - \gamma^2)\varphi + \beta_1\varphi^{\alpha_0+1} \\ \psi_t = \beta_{20} + \beta_2\varphi^{\alpha_0} \end{cases} \tag{63}$$

for the functions $\varphi(t)$ and $\psi(t)$. This system can be integrated and its general solution has the form

$$\begin{cases} \varphi(t) = \left[\frac{\gamma^2-\beta_{10}}{\beta_1+c_0\exp[(\gamma^2-\beta_{10})\alpha_0 t]}\right]^{1/\alpha_0} \qquad c_0 \in \mathbb{R} \\ \psi(t) = \left(\beta_{20} + \frac{\beta_2}{\beta_1}(\gamma^2 - \beta_{10})\right)t - \frac{\beta_2}{\beta_1\alpha_0}\ln|\beta_1 + c_0\exp[(\gamma^2 - \beta_{10})\alpha_0 t]|. \end{cases} \tag{64}$$

So, substituting (64) into ansatz (62) and taking into account (55), we arrive at an exact solution of the $\lambda - \omega$ system (54):

$$\begin{cases} U(t,x) = \left[\frac{\gamma^2-\beta_{10}}{\beta_1+c_0\exp[(\gamma^2-\beta_{10})\alpha_0 t]}\right]^{\frac{1}{\alpha_0}} \\ \quad \times\cos\left[\gamma x + \gamma_0 t - \frac{\beta_2}{\beta_1\alpha_0}\ln|\beta_1 + c_0\exp[(\gamma^2 - \beta_{10})\alpha_0 t]|\right] \\ V(t,x) = \left[\frac{\gamma^2-\beta_{10}}{\beta_1+c_0\exp[(\gamma^2-\beta_{10})\alpha_0 t]}\right]^{\frac{1}{\alpha_0}} \\ \quad \times\sin\left[\gamma x + \gamma_0 t - \frac{\beta_2}{\beta_1\alpha_0}\ln|\beta_1 + c_0\exp[(\gamma^2 - \beta_{10})\alpha_0 t]|\right] \end{cases} \tag{65}$$

where $c_0, \gamma \in \mathbb{R}$, $\gamma_0 = \beta_{20} + \frac{\beta_2}{\beta_1}(\gamma^2 - \beta_{10})$, $\gamma^2 - \beta_{10} \neq 0$, $\beta_1 \neq 0$. In the case $\beta_1 = 0$, the solution

$$\begin{cases} U(t,x) = c_0\exp[(\beta_{10} - \gamma^2)t]\cos\left[\gamma x + \beta_{20}t + \frac{\beta_2 c_0^{\alpha_0}}{\alpha_0(\beta_{10}-\gamma^2)}\exp[\alpha_0(\beta_{10} - \gamma^2)t]\right] \\ V(t,x) = c_0\exp[(\beta_{10} - \gamma^2)t]\sin\left[\gamma x + \beta_{20}t + \frac{\beta_2 c_0^{\alpha_0}}{\alpha_0(\beta_{10}-\gamma^2)}\exp[\alpha_0(\beta_{10} - \gamma^2)t]\right] \end{cases} \tag{66}$$

is obtained. Finally, the case $\gamma^2 - \beta_{10} = 0$ leads to the solution

$$\begin{cases} U(t,x) = [\alpha_0\beta_1(t_0 - t)]^{-\frac{1}{\alpha_0}}\cos\left[\sqrt{\beta}_{10}x + \beta_{20}t - \frac{\beta_2}{\beta_1\alpha_0}\ln|t_0 - t|\right] \\ V(t,x) = [\alpha_0\beta_1(t_0 - t)]^{-\frac{1}{\alpha_0}}\sin\left[\sqrt{\beta}_{10}x + \beta_{20}t - \frac{\beta_2}{\beta_1\alpha_0}\ln|t_0 - t|\right] \end{cases} \tag{67}$$

where $t_0 \in \mathbb{R}$.

The algebra $X_2 = P_t + vP_x + \gamma\partial_W$ generates the ansatz

$$\rho = \varphi(z) \qquad W = \psi(z) + \gamma t \qquad z = x - vt. \tag{68}$$

In this case the reduced ODE system for the functions $\varphi(z)$ and $\psi(z)$ has the form

$$\begin{cases} \varphi_{zz} - \varphi\psi_z^2 + v\varphi_z + \beta_{10}\varphi + \beta_1\varphi^{\alpha_0+1} = 0 \\ \varphi\psi_{zz} + 2\varphi_z\psi_z + v\varphi\psi_z + (\beta_{20} - \gamma)\varphi + \beta_2\varphi^{\alpha_0+1} = 0. \end{cases} \tag{69}$$

The system (69) is not integrable for arbitrary coefficients $v, \beta_k, \beta_{k0}, k = 1, 2$. However, one can obtain

$$\psi_z = c_0\varphi^{-2} \qquad c_0 \in \mathbb{R} \tag{70}$$

from the second equation if $\gamma = \beta_{20}, \beta_2 = v = 0$. Substituting (70) into the first equation of system (69), we obtain

$$x_0 \pm x = \int \frac{\varphi\,\mathrm{d}\varphi}{\sqrt{\frac{2\beta_1}{\alpha_0+2}\varphi^{\alpha_0+4} + c_1\varphi^2 - c_0^2}} \equiv I_{\alpha_0}(\varphi) \tag{71}$$

if $\alpha_0 \neq -2$, and

$$x_0 \pm x = \int \frac{\varphi\, \mathrm{d}\varphi}{\sqrt{2\beta_1\varphi^2 \ln\varphi + c_1\varphi^2 - c_0^2}} \equiv I_{\alpha_0}(\varphi)|_{\alpha_0=-2} \tag{72}$$

if $\alpha_0 = -2$.

So, assuming the existence of an inverse function $I_{\alpha_0}^{-1}$ to the I_{α_0} (see (71) and (72)) and using (68) with $\gamma = \beta_{20}$, $v = 0$ and (55), we arrive at an exact solution of the $\lambda - \omega$ system (54)

$$\begin{cases} U = I_{\alpha_0}^{-1}(x_0 \pm x)\cos(\psi(x) + \beta_{20}t) \\ V = I_{\alpha_0}^{-1}(x_0 \pm x)\sin(\psi(x) + \beta_{20}t) \end{cases} \tag{73}$$

where

$$\psi(x) = c_0 \int \frac{\mathrm{d}x}{\left[I_{\alpha_0}^{-1}(x_0 \pm x)\right]^2} \qquad x_0, c_0 \in \mathbb{R}.$$

It should be noted the special case $\varphi = U_0 = \text{constant}$, $\psi = \text{constant}$ leads to

$$U = U_0\cos\left(\sqrt{\lambda(U_0)}x + \omega(U_0)t\right) \qquad V = U_0\sin\left(\sqrt{\lambda(U_0)}x + \omega(U_0)t\right) \tag{74}$$

as a solution of (54), where $\lambda(U_0) = \beta_1 U_0^{\alpha_0} + \beta_{10}$, $\omega(U_0) = \beta_2 U_0^{\alpha_0} + \beta_{20}$. The periodic plane wave solution (74) is valid for arbitrary functions λ and ω [2].

Consideration of the algebra generated by operator X_3 does not lead to any invariant solutions because solving the relevant Lagrange system we obtain only the empty statement $\rho = \rho(t, x)$ and not an ansatz involving W.

Consider the algebra generated by the operator X_4; see (61). The relevant ansatz is

$$\rho = t^{-1/\alpha_0}\varphi(z) \qquad W = \psi(z) + \beta_{20}t + \gamma\ln t \qquad z = x/\sqrt{t} \tag{75}$$

which leads to the ODE system

$$\begin{cases} \varphi_{zz} - \varphi\psi_z^2 + \frac{z}{2}\varphi_z + \frac{1}{\alpha_0}\varphi + \beta_1\varphi^{\alpha_0+1} = 0 \\ \varphi\psi_{zz} + 2\varphi_z\psi_z + \frac{z}{2}\varphi\psi_z - \gamma\varphi + \beta_2\varphi^{\alpha_0+1} = 0. \end{cases} \tag{76}$$

Remark 6. In the case of the operator X_4^0, given by (61), one obtains (75) and (76) with $\beta_{20} = 0$, therefore it is a particular case of the case under consideration.

Unfortunately, the ODE system (76) is not integrable for arbitrary coefficients $\gamma, \alpha_0, \beta_k, k = 1, 2$ but a particular solution for $\gamma = \beta_2 = 0$, $\beta_1 \neq 0$ can be found, namely

$$\varphi = \left[-\frac{2(2+\alpha_0)}{\beta_1\alpha_0^2 z^2}\right]^{\frac{1}{\alpha_0}} \qquad \psi = -\beta_{20}t_0 \qquad t_0 \in \mathbb{R}. \tag{77}$$

Substituting (77) into the ansatz (75) we obtain as a solution of (54)

$$\begin{cases} U = \left[-\frac{2(2+\alpha_0)}{\beta_1\alpha_0^2 x^2}\right]^{\frac{1}{\alpha_0}}\cos(\beta_{20}(t - t_0)) \\ V = \left[-\frac{2(2+\alpha_0)}{\beta_1\alpha_0^2 x^2}\right]^{\frac{1}{\alpha_0}}\sin(\beta_{20}(t - t_0)). \end{cases} \tag{78}$$

Finally, we note that the following formula for the generation of new solutions for $\beta_{10} = 0$

$$\begin{cases} \rho_{new} = \epsilon^{2/\alpha_0}\rho^0(\epsilon^2(t - t_0), \epsilon(x - x_0)) \\ W_{new} = W^0(\epsilon^2(t - t_0), \epsilon(x - x_0)) + \beta_{20}(1 - \epsilon^2)t + c_0 \end{cases} \tag{79}$$

can be constructed by successive application of continuous transformations generated by the operators (57). Here $(\rho^0(t, x), W^0(t, x))$ is an arbitrary solution of (56) with $\beta_{10} = 0$ and $\epsilon \neq 0$, t_0, x_0 and c_0 are arbitrary parameters.

4. Conclusions

The theorems outlined in [1] and above give a complete description of Lie symmetries (i.e. a full group classification) of nonlinear multidimensional RD systems of the form (1). It has been established that there are three types of systems of the form (1) leading to essentially different Lie symmetries, namely: (a) $\lambda_1 \neq \lambda_2, \lambda_1\lambda_2 \neq 0$; (b) $\lambda_1 \neq 0, \lambda_2 = 0$; (c) $\lambda_1 = \lambda_2 \neq 0$.

Sets of local substitutions (namely (19) in [1], (21) in [1] and (7) above) have been found that reduce any other system with a non-trivial Lie symmetry to the corresponding system listed in tables 1–6 of [1] and tables 1–4 above.

In contrast to the scalar case, providing *a complete description* of local substitutions that reduce a given system of the form (1) to its most simplified (canonical) form is a very difficult problem, which we shall treat elsewhere. Here we note only that the sets of substitutions listed above and in [1] are simple in structure and very useful in simplifying several subclasses of systems of the form (1). For example, in each of the systems

$$\begin{cases} U_t = \Delta U \\ V_t = \Delta V + g(U) + \beta_{12} U \end{cases} \tag{80}$$

$$\begin{cases} U_t = \Delta U + U f(V) + \beta_{12} U \\ V_t = \Delta V + g(V) \end{cases} \tag{81}$$

and

$$\begin{cases} U_t = \Delta U + \beta_1 U \qquad \beta_1 \neq \beta_2 \\ V_t = \Delta V + \beta_2 V + g(U) + \beta_{12} U \end{cases} \tag{82}$$

containing arbitrary smooth functions f and g, we can set $\beta_{12} = 0$ by the substitutions

$$\begin{cases} U \to U \\ V \to V + \beta_{21} t U \end{cases} \tag{83}$$

$$\begin{cases} U \to U \exp(\beta_{21} t) \\ V \to V \end{cases} \tag{84}$$

and

$$\begin{cases} U \to U \\ V \to V + \frac{\beta_{21}}{\beta_1 - \beta_2} U \end{cases} \tag{85}$$

respectively. Substitutions (83)–(85) are particular cases of (7).

To give an example of a very non-trivial local substitution (not belonging to the classes mentioned so far), let us consider the system arising in case 19, table 4. We have found the substitution

$$\begin{cases} U \to U + \beta_1 t \\ V \to V + \beta_1 \beta_{20} t^2 U + \frac{1}{3}\beta_1^2 \beta_{20} t^3 \end{cases} \tag{86}$$

that reduces the system to one with $\beta_1 = 0$. In other words, case 19, table 4 is reducible to case 17, table 4 using (86). Another new local substitution follows from remark 5.

Having now a complete description of Lie symmetries, we have established that several RD systems arising in various applications admit non-trivial symmetries. Two of them, a limiting case (7) in [1] of a model used to describe a biological pattern arising in hydra and the $\lambda - \omega$ system (54) have been considered in detail. All non-equivalent Lie ansätze, formulae for mapping between solutions and examples of non-trivial exact solutions have been constructed and some of their properties investigated.

It seems worthwhile to compare our results with those obtained in recently published paper [7] where Lie symmetries of reaction–diffusion systems with cross-diffusion

$$\begin{cases} U_t = a_{11}\Delta U + a_{12}\Delta V + F(U, V) \\ V_t = a_{21}\Delta U + a_{22}\Delta V + G(U, V) \end{cases} \tag{87}$$

are described (here $a_{ij} \in \mathbb{R}, i = 1, 2 j = 1, 2$).

As we noted in [1], our method of Lie symmetry classification is based on the classical Lie scheme and on finding and then making essential use of the sets of local substitutions that reduce any system with a non-trivial Lie symmetry to one given in the relevant tables. On the other hand, it is shown in [7] that the Lie scheme for the system (87) admits a formulation in terms of commutator algebras, so the authors adopt a different approach, though, as explicitly noted in [7], these substitutions were not used systematically.

The first (a) and third (c) (see [1], p 270) types of system (1) are evidently particular cases of (87) with $a_{12} = a_{21} = 0$. However, the second type (b) (see [1], p 270) cannot be obtained from (87).

In the case (a), the authors of [7] have compared their results with those obtained in [1]. However, account was not taken of table 6 in [1] so their comments by way of comparison are valid only in a particular case (that noted in remark 3 above). Although tables II–IV in [7], representing the main results, are somewhat cumbersome (the nonlinearities and the relevant Lie symmetries can be essentially simplified) and contain misprints (see also (i)–(iii) below), we have attempted to compare the results in the case (c) too, i.e. $a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$ in the system (87). We have established the following conclusions with regard to this special case.

(i) All nonlinearities listed in tables II–IV [7] can be found either in tables 1–4 or in remarks 2 and 3. It should be stressed that the sets of local substitutions (19) in [1], (21) in [1] and (7) above have to be exploited in showing this.

(ii) Several additional Lie symmetry operators are missing in tables III and IV of [7]. Indeed, comparing the last subcase of case 4 in table III of [7] with the relevant case, 20 (see table 3 above), one observes the omission of the analogue of the operator Q_α. Similarly, analogues of the operators $Q + \gamma t U \partial_V$ (see case 22 with $\beta_{10} = 0$ in table 3), R^1_α (see case 4 in table 4) and $\partial_U + V\partial_V$ (see case 22 in table 4) are missing in table III of [7] (see the second subcase of case 5, case 9, and the last subcase of case 7, respectively). Analogues of the operator $\mathcal{Q}^1_0$ (see cases 15 and 18 in table 3) are also missing in table IV of [7] (see the last two subcases and the first two subcases of case 2).

(iii) There are several types of nonlinearities leading to non-trivial Lie symmetries in addition to those in [7]. Case 3 from table 4 needs to be included into table II of [7]. Note that the relevant system contains two arbitrary functions. Similarly, cases 1 and 2 from table 1, cases 19, 21 and 26 from table 3 and cases 5, 9, 16, 17, 19 and 24 from table 4 need to be added to table III of [7] as new cases. One can also find relevant cases in tables 1–4 which are absent in table IV of [7] (for example, cases 6–8 and 10–14 from tables 4).

It should be stressed that a complete description of Lie symmetries of a *nonlinear PDE system* containing *arbitrary functions of two or more dependent variables* is a quite difficult task. The pioneering works were published only a few years ago. To our knowledge the first one is the paper [9], where the authors have solved this problem in the case of a nonlinear system containing a two-dimensional PDE and an ODE, while the paper [1] is the first involving nonlinear systems of multidimensional PDEs.

Finally, we would like to discuss briefly the next steps in a Lie symmetry description of nonlinear multidimensional systems containing arbitrary functions. In the case $\lambda_1 = \lambda_2 = 0$

the reaction–diffusion system (1) degenerates into an elliptic system. The complete description of Lie symmetries for the corresponding two-dimensional elliptic system (i.e. $U = U(x_1, x_2)$, $V = V(x_1, x_2)$) was performed in [10] and work is in progress for the higher dimensional case. Similarly, work is in progress for the case of the variable diffusivities D_1 and D_2, i.e. for reaction–diffusion systems of the form

$$\begin{cases} U_t = \left(D_1(U, V)U_{x_a}\right)_{x_a} + F(U, V) \\ V_t = \left(D_2(U, V)V_{x_a}\right)_{x_a} + G(U, V) \end{cases}$$

where summation is assumed from 1 to n over the repeated indices a.

It also seems reasonable to seek a complete Lie symmetry description of nonlinear reaction–diffusion systems containing three or more equations for unknown functions $U_1(t, x), U_2(t, x), \ldots, U_m(t, x)$, $m \geqslant 3$. Our experience accumulated in the cases $m = 1$ [11] and $m = 2$ [1] suggests that this is an extremely difficult problem which will be solved only by the development of a special computer algebra package and its subsequent application, together with a relevant modification of the classical Lie scheme. Some preliminary investigations indicate that about 10^3 non-equivalent systems with non-trivial Lie symmetries arise in the case $m = 3$.

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